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Soliton Dynamics in Computational Anatomy

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Summary

Solitons are solutions of certain evolutionary systems governed by nonlinear partial differential equations (PDE). The pursuit of these solutions has produced many exciting new ideas in mathematics and physics. These ideas turned out to be universal, leading to a wide range of applications. The pursuit of these ideas has revealed a fundamental and powerful geometrical framework for the investigation of the solution behavior of nonlinear evolutionary PDEs. Here we explain how the ideas of soliton theory apply to a leading approach in image analysis called template matching. Namely, a certain soliton equation (EPDiff) and its special property (its momentum map) hold the keys for optimizing the template matching procedure and achieving the three principle goals of an emerging new science called Computational Anatomy.

Anatomical features revealed by modern imagery are recognized in computational anatomy (CA), by comparisons of shapes. The basic CA technique for comparing shape is called "template matching" (TM). The first goal of CA is to develop a comparison process for template matching which would automatically select optimal paths of transformations depending continuously on a parameter and mapping between the shapes or images. For this, one would need to write an equation whose solution would flow along the optimal path in the transformation group leading from one shape to the other.

The equation needed for CA was discovered about a decade ago at Los Alamos by Camassa and Holm [1]. It emerged unexpectedly in the dynamics of ***solitons*** describing the evolution of nonlinear shallow water waves.

How do nonlinear shallow water waves apply to the study of shapes of images? Just think of the crests of the waves as the outlines, or "cartoons" of the images. Now let the moving crests of the waves represent the deformation of the outlines of the image. Continuing our theme, it turns out the waves described by the Camassa-Holm equation [1] evolve optimally along a curve in the group of smooth invertible transformations of shape called the diffeomorphisms (or "diffeos," for short). The diffeo group contains the conformal group but acts in any number of spatial dimensions, not just the plane. In the diffeos the optimal path from one shape to another is found by letting one of the two shapes be the reference shape and taking the other one as the target shape, as an optimal control problem. In addition, once freed from their context as shallow water waves in two dimensions, the application of diffeos for transforming images allows the shapes on which they act

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to be compared in three dimensions.

The necessary equation in three dimensions is a generalization of the Camassa-Holm equation which is now called EPDiff [2]. (Of course, Diff stands for "diffeos.")

EPDiff has several key mathematical properties which allow progress toward achieving the three principal goals of CA.
(i) The first goal of automated construction of shapes is fulfilled in principle by showing that the solutions of EPDiff encode the properties required to follow the growth and form of anatomical structures.
(ii) Solutions of EPDiff describe optimal smooth paths (called geodesics) in the diffeos. So they provide the appropriate framework for achieving the second goal of CA, namely, optimal comparisons.
(iii) Finally, singular solutions of EPDiff exist and these may be parameterized using linear spaces. This property follows from the mathematics of the "momentum map." The momentum map describes a class of singular solutions that are precisely the required transformations of subspaces which form anatomical configurations, by characterizing the paths of points, curves, surfaces and subvolumes. These are called "landmarks" in the CA community.

Here is where the properties of nonlinear water waves re-enter the picture. Water waves have **momentum** which is exchanged when they collide. *Do images also have momentum?* Identifying their outlines and landmarks as the analog of solitons using the momentum map provided the evolving landmarks in the comparison of images with characteristics of both a position and a momentum, which both live in linear vector spaces. This produces a complete non-redundent specification of the landmarks of the image and their evolution in a representation that being linear is amenable to both statistical studies and error analysis. Being solutions of EPDiff ensures that they transform along the optimal path from one image to another. This confluence of ideas has only recently emerged, and much work remains to be done to achieve its goals, both in the theory and in the applications of these soliton ideas to the comparison and transformation of images. For a recent description in more technical detail of the ideas based on soliton theory and a bibliography of key papers in the field of CA, see [3].

References:

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